

## Suprathreshold stochastic resonance and noise-enhanced Fisher information in arrays of threshold devices

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We analyze the parametric estimation that can be performed on a signal buried in noise based on the parsimonious representation provided by a parallel array of threshold devices. The Fisher information contained in the array output about the input parameter is used as the measure of performance in the estimation task. For estimation on a suprathreshold input signal, we establish that enhancement of the Fisher information can be obtained by addition of independent noises to the thresholds in the array. Similar improvement by noise is also shown to be possible for the estimation error of the maximum likelihood estimator. These results extend the applicability of the recently introduced nonlinear phenomenon of suprathreshold stochastic resonance.

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### I. INTRODUCTION

Stochastic resonance (SR) is a nonlinear phenomenon which was introduced some twenty years ago and which describes the possibility of improvement of the transmission or the processing of a signal, thanks to the action of the noise [1]. Since its introduction, SR has gradually been shown feasible under many different forms, with various types of signals, nonlinear processes, and measures of performance receiving improvement from the noise [2–10]. Most occurrences of SR involve a signal which is by itself too small or ill conditioned to elicit a strong response from a nonlinear system. Addition of noise then brings assistance to the small signal in eliciting a more efficient response from the nonlinear system, for instance by overcoming a threshold or a potential barrier. Very recently, another interesting form of SR has been introduced under the name of suprathreshold SR [11,12]. In this case, SR is obtained in the response of a parallel array of threshold devices. The input signal by itself is strong enough to overcome the threshold of a single device, and it needs no assistance from noise for this. With no noise in the array, the input signal usually elicits the same output response from any one of the devices. If different noises are independently added on the devices of the array, then each device will in general produce a distinct response. When all these responses are collected over the array, it is shown in Refs. [11,12] that the global response can be more efficient than the response of a single device with no noise. Furthermore, an optimal nonzero amount of noise can be found that maximizes the efficacy of the global response of the array. This suprathreshold SR, as introduced in Ref. [11], is important since it significantly extends the mechanisms under which SR, as an improvement by noise, can occur.

Suprathreshold SR in Refs. [11,12] is observed and measured by means of the input-output mutual information across the array. It is shown in Refs. [11,12], for the transmission of a suprathreshold input signal, that a maximum mutual information is obtainable for a nonzero amount of noise added to the thresholds. Here, in the present paper, we propose further explorations of the phenomenon of suprathreshold SR. We consider the same type of parallel array of

threshold devices as in Refs. [11,12], but we use this array for a markedly distinct information processing operation. The array is used here for estimation of a signal parameter. To assess the performance of the array in the estimation task and to investigate the possibility of suprathreshold SR, we rely on, instead of the mutual information, the Fisher information, which is specifically relevant to quantify the efficacy of estimation processes. Applications of the Fisher information to quantify standard (subthreshold) SR have been proposed in Refs. [13–15]. In those studies, a single threshold device along with a subthreshold input signal is considered. It is shown that noise can help the subthreshold input to overcome a single threshold, this being reflected by the possibility of a noise-enhanced input-output Fisher information. Here, in the present paper, we examine the behavior of the input-output Fisher information in the transmission of a suprathreshold input signal across the parallel array. We establish that noise enhancement of the Fisher information can be obtained, demonstrating a distinct manifestation of a suprathreshold SR phenomenon. The associated possibility of improvement by the noise is also exhibited in the performance of the maximum likelihood estimator from the array output.

### II. NONLINEAR TRANSMISSION IN A PARALLEL ARRAY

A random signal  $x(t)$  is dependent upon an unknown parameter  $a$ , the value of which we seek to estimate. The measurements on the signal  $x(t)$  are obtained by means of threshold devices or one-bit quantizers. We consider, as in Refs. [11,12], a parallel array of a number  $N$  of such one-bit quantizers. A noise  $\eta_i(t)$ , independent of  $x(t)$ , can be added to  $x(t)$  before quantization by quantizer  $i$ . Quantizer  $i$ , with threshold  $\theta_i$ , delivers the output

$$y_i(t) = H[x(t) + \eta_i(t) - \theta_i], \quad i = 1, 2, \dots, N, \quad (1)$$

where  $H(u)$  is the Heaviside function, i.e.,  $H(u) = 1$  if  $u > 0$  and is zero otherwise. We will consider here that the  $N$  noises  $\eta_i(t)$  are white, mutually independent, and identically distributed with cumulative distribution function  $F_{\eta}(u)$  and

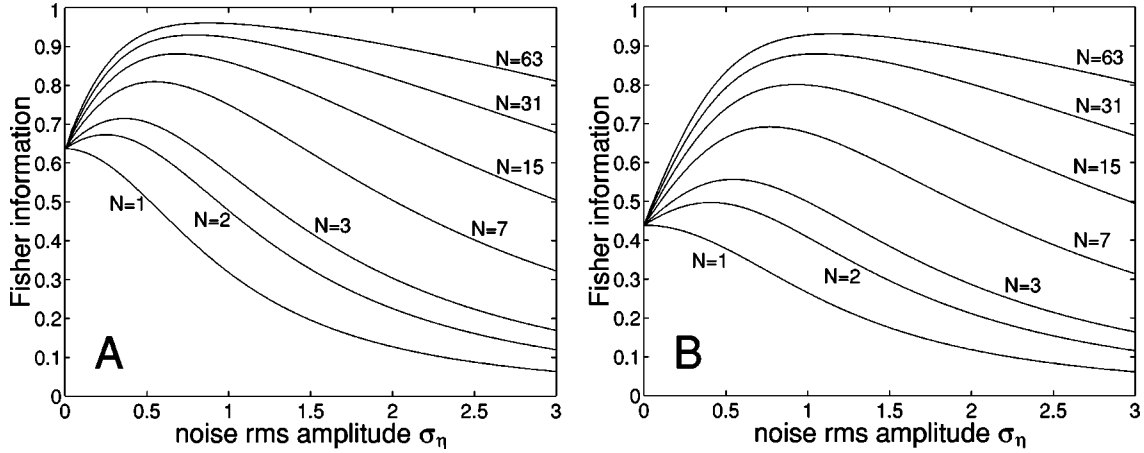


FIG. 1. Fisher information  $J_Y$  of Eq. (3), as a function of the rms amplitude  $\sigma_\eta$  of the array noises  $\eta_i(t)$  chosen zero-mean Gaussian. The input random signal  $x(t)$  is Gaussian with mean  $a$  and standard deviation  $\sigma_x = 1$ . All thresholds in the array are set to  $\theta = 0$ . Panel A, when  $a = 0$  and panel B, when  $a = 1$ .

probability density function  $f_\eta(u) = dF_\eta/du$ . The response  $Y(t)$  of the array is obtained by summing the outputs of all the quantizers as

$$Y(t) = \sum_{i=1}^N y_i(t). \quad (2)$$

The estimation of the parameter  $a$  is to be based on the observation of  $Y(t)$  alone. Such conditions, with arrays of one-bit quantizers, are specially relevant for existing and future multisensor networks having to cope with limited time and resources for data processing, storage, communication, and for energy supply.

For the estimation of  $a$  from  $Y(t)$ , a key quantity [16] is the Fisher information  $J_Y$  contained in  $Y(t)$  about  $a$ . Fisher information  $J_Y$ , via the Cramér-Rao inequality, sets a bound to the efficacy of any conceivable unbiased estimator of  $a$  from  $Y(t)$ : the variance of any such estimator is lower bounded by the reciprocal of the Fisher information. For  $Y(t)$ , which assumes integer values between 0 and  $N$ , the Fisher information  $J_Y$  is

$$J_Y = \sum_{n=0}^N \frac{1}{\Pr\{Y(t) = n\}} \left[ \frac{\partial}{\partial a} \Pr\{Y(t) = n\} \right]^2. \quad (3)$$

At time  $t$ , for a fixed given value  $x$  of the input signal  $x(t)$ , we have the conditional probability  $\Pr\{y_i(t) = 0 | x\}$ , which is also  $\Pr\{x + \eta_i(t) \leq \theta_i\}$ , amounting to

$$\Pr\{y_i(t) = 0 | x\} = F_{\eta_i}(\theta_i - x) = q_i(x). \quad (4)$$

In the same way, we have  $\Pr\{y_i(t) = 1 | x\} = 1 - q_i(x)$ .

We assume for the present, as done in Ref. [12], that all the thresholds  $\theta_i$  share the same value  $\theta_i = \theta$  for all  $i$ , and  $F_{\eta_i}(\theta - x) = q(x)$ . The conditional probability  $\Pr\{Y(t) = n | x\}$  then follows, according to the binomial distribution, as

$$\Pr\{Y(t) = n | x\} = C_n^N [1 - q(x)]^n q(x)^{N-n}, \quad (5)$$

where  $C_n^N$  is the binomial coefficient. We therefore obtain the probability

$$\Pr\{Y(t) = n\} = \int_{-\infty}^{+\infty} C_n^N [1 - q(x)]^n q(x)^{N-n} f_x(x) dx, \quad (6)$$

where  $f_x(u)$  is the probability density function of the input signal  $x(t)$ .

For the sake of definiteness, concerning the parametric dependence of  $x(t)$  on  $a$ , we shall consider in the sequel the broad class of processes where  $x(t)$  is formed by the additive mixture  $x(t) = \xi(t) + s_a(t)$ . The signal  $\xi(t)$  is a random (native) noise, white, independent of the  $\eta_i$ 's and of  $a$ , with probability density  $f_\xi(u)$ . The signal  $s_a(t)$  is deterministic and contains the parameter  $a$ . For instance,  $a$  can be the value of a constant  $s_a(t) \equiv a$ , the amplitude or frequency of a periodic  $s_a(t)$ , or any other parameter entering the specification of the deterministic  $s_a(t)$ . We then have for the density  $f_x(u) = f_\xi[u - s_a(t)]$  and for the derivative of Eq. (6) with respect to  $a$ ,

$$\begin{aligned} \frac{\partial}{\partial a} \Pr\{Y(t) = n\} \\ = - \frac{\partial s_a(t)}{\partial a} \int_{-\infty}^{+\infty} C_n^N [1 - q(x)]^n q(x)^{N-n} f'_\xi[x - s_a(t)] dx. \end{aligned} \quad (7)$$

The Fisher information  $J_Y$  of Eq. (3) follows directly from Eqs. (6) and (7), possibly through numerical integration, in broad conditions concerning the noises  $\eta_i(t)$  and the input signal  $x(t)$ .

For illustration of the possibility of a suprathreshold SR measured by  $J_Y$ , we consider the case where  $s_a(t)$  is the constant signal  $s_a(t) \equiv a$  and  $\xi(t)$  is a zero-mean noise. Therefore, the input signal is  $x(t) = a + \xi(t)$  and our estimation task amounts to estimating the mean  $a$  of the random signal  $x(t)$ , whose standard deviation is denoted by  $\sigma_x$  [it is also the standard deviation of  $\xi(t)$ ]. Figure 1 shows evolu-

tions of the resulting Fisher information  $J_Y$  of Eq. (3), as a function of the rms amplitude  $\sigma_\eta$  of the array noises  $\eta_i(t)$ , in some typical conditions.

It is remarkable that the evolutions of the Fisher information in Fig. 1 are quite similar to those of the Shannon mutual information presented in Refs. [11,12], although these two information measures are *a priori* quite distinct. In Fig. 1, the input signal  $x(t)$  is always suprathreshold. At  $N=1$ , with a single threshold, addition of the threshold noise  $\eta_1(t)$  always degrades the Fisher information  $J_Y$ , much like in Ref. [11] with the Shannon information. For  $N>1$ , with no added noises  $\eta_i(t)$  on the thresholds, all the quantizers switch in unison and the array acts just like a single one-bit quantizer. It is when the threshold noises  $\eta_i(t)$  are added that the quantizer outputs  $y_i$  start to behave differently for different  $i$ , giving access to a richer representation of the suprathreshold input  $x(t)$ . This is conveyed in Fig. 1 by a Fisher information  $J_Y$  which increases when the level  $\sigma_\eta$  of the threshold noises  $\eta_i(t)$  grows, up to an optimal nonzero  $\sigma_\eta$  where  $J_Y$  is maximized. For increasing  $N$ , this maximum Fisher information also increases. This maximum of  $J_Y$  tends to reach  $1/\sigma_x^2$  which represents, in the conditions of Fig. 1 with a Gaussian  $x(t)$ , the input Fisher information contained in the analog input signal  $x(t)$  about the parameter  $a$ . Again, at  $N>1$ , the behavior of the Fisher information  $J_Y$  in presence of the threshold noises  $\eta_i(t)$  is quite reminiscent of the behavior of the Shannon information in Refs. [11,12]. These observations tend to prove that suprathreshold SR, as introduced in Ref. [11], much like standard (subthreshold) SR (although the mechanism is different), is a general nonlinear phenomenon which can occur and be quantified in many different ways. It expresses that an array of nonlinear devices in charge of the transmission of a suprathreshold signal will be more efficient if the devices of the array are allowed to respond in a nonuniform way, thanks to the addition of independent noises on the devices, with an efficiency which can be assessed in *a priori* many different ways.

Figure 1 shows the Fisher information  $J_Y$  from a single measurement at the output of the array of  $N$  devices. The Fisher information from  $M$  independent measurements would be  $MJ_Y$ . It is then observable from the curves of Fig. 1 that  $N$  independent measurements from a single device would in general contain more Fisher information than one measurement from an array of  $N$  devices. The benefit with the array is the quasi-instantaneous character of a single measurement, while  $N$  measurements require much longer time. The array will allow a much larger repetition rate for successive estimation tasks. The picture is similar with the Shannon information of Refs. [11,12].

The evolutions of Fig. 1 have been obtained through numerical evaluation of the integrals of Eqs. (6) and (7). Numerical integration allowed us here to demonstrate the suprathreshold SR in the Fisher information for the practically very important case of Gaussian noises  $\xi(t)$  and  $\eta_i(t)$ , much like the demonstration of the suprathreshold SR in the Shannon information was obtained in Refs. [11,12]. Later on, for the suprathreshold SR in the Shannon information, complementary studies appeared that found special configurations allowing one to push further the analytical treatment, con-

firmed the effect in other special cases and bringing additional insight to the description [17,18]. In a similar perspective here, for the suprathreshold SR in the Fisher information, we address in the Appendix the case of uniform noises  $\xi(t)$  and  $\eta_i(t)$ , these conditions allowing us to push further the analytical description.

### III. DISTRIBUTION OF THRESHOLDS

When the thresholds  $\theta_i$ ,  $i=1$  to  $N$ , no longer share the same value  $\theta$ , the conditional probability  $\Pr\{Y(t)=n|x\}$  of Eq. (5) has to be computed as

$$\Pr\{Y(t)=n|x\} = \sum_{(n)} \prod_{i=1}^N [1 - q_i(x)]^{y_i} q_i(x)^{1-y_i}, \quad (8)$$

where  $\sum_{(n)}$  stands for the sum over the states available to  $Y(t)$  for which the number of  $y_i$  equal to 1 is exactly  $n$ , among the  $2^N$  distinct states available to  $Y(t)$ . After this replacement of Eq. (5) by Eq. (8) is done, the probability  $\Pr\{Y(t)=n\}$  follows in the same way as

$$\Pr\{Y(t)=n\} = \int_{-\infty}^{+\infty} \Pr\{Y(t)=n|x\} f_x(x) dx \quad (9)$$

and its derivative with  $a$  when  $x(t) = \xi(t) + s_a(t)$  and  $f_x(u) = f_\xi[u - s_a(t)]$ , as

$$\begin{aligned} & \frac{\partial}{\partial a} \Pr\{Y(t)=n\} \\ &= - \frac{\partial s_a(t)}{\partial a} \int_{-\infty}^{+\infty} \Pr\{Y(t)=n|x\} f'_\xi[x - s_a(t)] dx, \end{aligned} \quad (10)$$

providing access to  $J_Y$  of Eq. (3).

Establishing the exact distribution of the thresholds  $\theta_i$ , which maximizes the Fisher information  $J_Y$  of Eq. (3), is a complicated problem, especially when  $N$  is not too small. We shall avoid this problem here, since our purpose is more focused on establishing the feasibility of a noise-enhanced Fisher information with arrays of quantizers in various reasonable conditions. Although not proven optimal in relation to the Fisher information, a reasonable choice for the thresholds  $\theta_i$  is the maximum-entropy distribution making all the output states equiprobable by imposing  $\int_{\theta_i}^{\theta_{i+1}} f_x(u) du = 1/(N+1)$ , as in Ref. [19]. When  $f_x(u)$  is zero-mean Gaussian with standard deviation  $\sigma_x$ , the thresholds follow as

$$\theta_i = \sigma_x \sqrt{2} \operatorname{erf}^{-1} \left( \frac{2i}{N+1} - 1 \right). \quad (11)$$

For such a distribution of the thresholds, Fig. 2 illustrates that a suprathreshold SR is still possible, in definite conditions. Figure 2 considers the case of the estimation of the mean  $a$  of a Gaussian random input  $x(t)$  with standard deviation  $\sigma_x = 1$ . We choose a number  $N=7$  of thresholds according to Eq. (11), especially yielding  $\theta_4=0$  and  $\theta_7 \approx 1.15$ . Figure 2 shows various evolutions of the Fisher in-

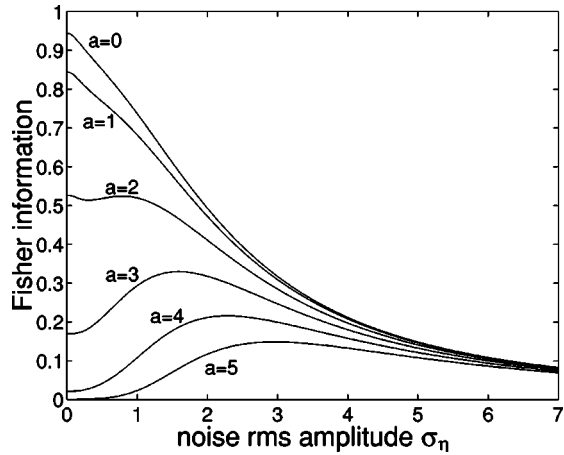


FIG. 2. Fisher information  $J_Y$  of Eq. (3), as a function of the rms amplitude  $\sigma_\eta$  of the array noises  $\eta_i(t)$  chosen zero-mean Gaussian. The input random signal  $x(t)$  is Gaussian with mean  $a$  and standard deviation  $\sigma_x=1$ . The thresholds  $\theta_i$  are set according to Eq. (11), for  $i=1$  to  $N=7$ .

formation  $J_Y$  of Eq. (3), for different values of the mean  $a$ . The value of  $a$  determines how the input  $x(t)$  is seen by the array of thresholds  $\theta_i$ . For any value of  $a$  in Fig. 2, the input  $x(t)$  is always suprathreshold. For  $a$  equal or close to zero (at the scale set by  $\sigma_x=1$ ), the input  $x(t)$  is well centered in relation to the array of thresholds. In such conditions the array noises  $\eta_i(t)$  are always detrimental, as expressed by the monotonic decay of the Fisher information  $J_Y$  in Fig. 2, as  $\sigma_\eta$  grows. By contrast, for larger  $a$  the input  $x(t)$  comes to be, on average, not well centered in relation to the array of thresholds. In such conditions, the array noises  $\eta_i(t)$  bring the possibility of some shift in the array of thresholds  $\theta_i$ , and this on average, tends to be beneficial to the transmission of the input  $x(t)$ . This is conveyed by a Fisher information  $J_Y$  in Fig. 2, which can be increased as the level  $\sigma_\eta$  of the array noises grows, with  $J_Y$  culminating at a maximum for an optimal nonzero noise level. This is an instance of suprathereshold SR, under the form of a noise-enhanced Fisher

information across an array of distributed thresholds. When a suprathreshold signal to be estimated is not well positioned in relation to the array of thresholds, noise addition to the thresholds can bring improvement to the performance in estimation.

Other conditions exist, allowing a suprathreshold SR in an array of distributed thresholds. For instance, if the thresholds are set according to Eq. (11) with  $\sigma_x=1$ , and subsequently the array is used to transmit an input  $x(t)$  with a different standard deviation  $\sigma_x$  (above or below 1), then a suprathreshold SR can occur in conditions qualitatively similar to those of Fig. 2. Varying  $N$  and the distribution of thresholds  $\theta_i$  (for instance, equispaced thresholds) also preserve the possibility of a suprathreshold SR.

#### IV. AVERAGE FISHER INFORMATION

A difficulty for drawing full benefit of SR for estimation is that in general the optimal level of the noise to be added is dependent upon the unknown value of the parameter  $a$  we seek to estimate. This is the same picture for the standard (subthreshold) SR for estimation considered in Refs. [13–15], as well as for the suprathreshold SR shown here in Figs. 1 and 2. In practice it is reasonable to admit that some prior knowledge is available concerning the possible values or range for the parameter  $a$  to be estimated. If this prior knowledge is expressible by a prior probability density  $f_a(u)$  for the possible values of  $a$ , then an optimal noise level can be determined to maximize the average, over this prior density, of the Fisher information  $J_Y$  of Eq. (3) seen as a function of  $a$ , i.e.,

$$\bar{J}_Y = \int_{-\infty}^{+\infty} J_Y(a) f_a(a) da. \quad (12)$$

For estimation purpose, this will guarantee that for a large number of values of  $a$ , drawn from  $f_a(u)$ , the average performance will be optimized.

Figure 3 illustrates the possibility, in various conditions,

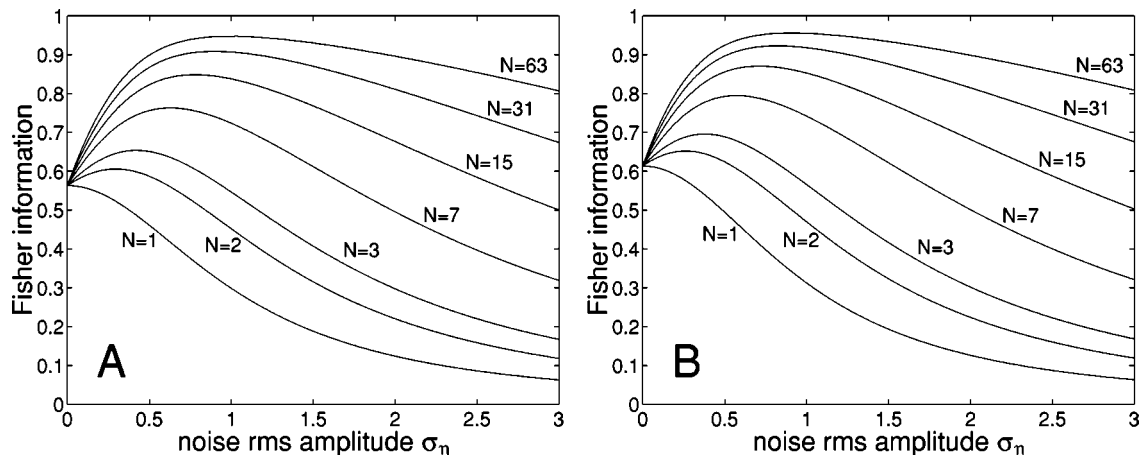


FIG. 3. Average Fisher information  $\bar{J}_Y$  of Eq. (12), as a function of the rms amplitude  $\sigma_\eta$  of the array noises  $\eta_i(t)$  chosen zero-mean Gaussian. The input random signal  $x(t)$  is Gaussian with mean  $a$  and standard deviation  $\sigma_x=1$ . The constant  $a$  is uniformly distributed over  $[0,1]$ . All thresholds in the array are set to  $\theta=0$  (panel A) and  $\theta=0.5$  (panel B).

of a suprathreshold SR on the average Fisher information  $\bar{J}_Y$  of Eq. (12), in the case where all the thresholds  $\theta_i$  share the same value  $\theta$ .

Curves like those of Fig. 3 can be used to *a priori* select the optimal level  $\sigma_\eta$  of the array noise  $\eta_i(t)$  that maximizes  $\bar{J}_Y$  of Eq. (12) for definite conditions of operation concerning  $f_a(u)$ ,  $N$ , and the common threshold  $\theta$ . Furthermore, the present treatment also allows us to determine the optimal value of the common threshold  $\theta$  to maximize  $\bar{J}_Y$  of Eq. (12) at the optimal noise level [in many conditions the outcome will be the expectation  $\theta = E(a)$ ].

Another distinct strategy to bring the array to operate at the optimal level of the noises  $\eta_i(t)$  for estimation of a given  $a$  is to adjust the noise level through an adaptive procedure. Adaptation strategies for subthreshold input have been briefly discussed in Ref. [13], and recently in more detail in Ref. [20], especially in the context of neuronal signal transmission aided by noise. An adaptive procedure is also used with standard (subthreshold) SR for estimation from a single one-bit quantizer in Ref. [21]. In the context of suprathreshold SR, the approach here would be to start with no array noises, at  $\sigma_\eta = 0$ . Get an estimate of  $a$ , which will not benefit from operation at the maximal Fisher information. Use calibration curves like those of Fig. 1 or 2, given  $N$  and the common  $\theta$  or the distribution of  $\theta_i$ , in order to deduce the optimal noise level corresponding to the current estimated value for  $a$ . Add array noises with this level, redo an estimation of  $a$ , and iterate the process. After a few iterations, this procedure will bring the array to operate in the vicinity of the optimal noise level  $\sigma_\eta$  associated to the maximum Fisher information  $J_Y$  of Eq. (3) for a given  $a$ .

Adaptation strategies could also be envisaged for the thresholds, but the possibility of threshold adjustment is not always easily available, for instance in neuronal contexts. Also, this means falls outside the scope of SR, which specifically aims at exploiting the noise to assist the signal, and whose potentialities we are investigating here.

## V. MAXIMUM LIKELIHOOD ESTIMATION

For actual estimation, a reasonable strategy is to collect from the output of the array a set of  $M$  measurements  $Y_j = Y(t_j)$  at  $M$  distinct times  $t_j$ , for  $j = 1$  to  $M$ , and to consider the estimation of  $a$  from the data set  $(Y_1, \dots, Y_M) = \mathbf{Y}$  by means of the maximum likelihood [22] estimator  $\hat{a}_{ML}$ . The likelihood  $L(\mathbf{Y}; a)$  is defined as the probability  $\Pr\{\mathbf{Y}; a\}$ , and since we are dealing with white noises, we have

$$L(\mathbf{Y}; a) = \Pr\{\mathbf{Y}; a\} = \prod_{j=1}^M \Pr\{Y_j; a\}. \quad (13)$$

For a given measurement  $Y_j$ , which takes on an integer value between 0 and  $N$ , the probability  $\Pr\{Y_j; a\}$  is the probability of Eq. (6) seen as a function of  $a$  at given  $Y_j$ , i.e.,

$$\Pr\{Y_j; a\} = \int_{-\infty}^{+\infty} C_{Y_j}^N [1 - q(x)]^{Y_j} q(x)^{N - Y_j} f_\xi[x - s_a(t)] dx. \quad (14)$$

The expression of Eq. (14) is for the case of a common threshold  $\theta$ ; for a distribution of thresholds  $\theta_i$  a comparable expression is accessible from Eqs. (8) and (9). The maximum likelihood estimator  $\hat{a}_{ML}$  is then defined as

$$\hat{a}_{ML} = \arg_a \max L(\mathbf{Y}; a). \quad (15)$$

The performance of the maximum likelihood estimator is naturally assessed by the rms estimation error  $\mathcal{E}$  defined through the expectation

$$\mathcal{E} = \sqrt{E[(a - \hat{a}_{ML})^2]}. \quad (16)$$

For estimation of  $a$  based on  $\mathbf{Y} = (Y_1, \dots, Y_M)$ , the Fisher information contained in  $\mathbf{Y}$  about  $a$  is  $MJ_Y$ . Any unbiased estimator of  $a$  from  $\mathbf{Y}$  will have its rms error lower bounded by the Cramér-Rao bound  $1/\sqrt{MJ_Y}$  derived from the Fisher information  $J_Y$  of Eq. (3). The maximum likelihood estimator  $\hat{a}_{ML}$  of Eq. (15), as the number of measurements  $M$  becomes large, is asymptotically unbiased and efficient [22], i.e., its rms error  $\mathcal{E}$  reaches (from above) the Cramér-Rao bound  $1/\sqrt{MJ_Y}$ .

We have performed a series of Monte Carlo trials for estimation on a suprathreshold signal  $x(t)$ , implementing a numerical evaluation of the maximum likelihood estimate from Eqs. (13)–(15). This has been done for a fixed constant signal  $s_a(t) \equiv a = a_{\text{true}}$ , in the following way. A vector realization of the native noise  $[\xi(t_1), \dots, \xi(t_M)]$  is generated, with each  $\xi(t_j)$ ,  $j = 1$  to  $M$ , independently drawn from the known density  $f_\xi(u)$ . For each signal sample  $x(t_j) = a_{\text{true}} + \xi(t_j)$ , the  $N$  realizations  $\eta_i(t_j)$ ,  $i = 1$  to  $N$ , of the threshold noises are obtained, with each  $\eta_i(t_j)$  independently drawn from the known density  $f_\eta(u)$ ; the corresponding  $y_i(t_j)$ ,  $i = 1$  to  $N$ , are deduced through Eq. (1). The value of  $Y(t_j) = Y_j$  is then deduced from Eq. (2). The  $M$  values of  $Y_j$ ,  $j = 1$  to  $M$ , are collected to provide a given realization of the measurement vector  $\mathbf{Y} = (Y_1, \dots, Y_M)$ . For this realization  $\mathbf{Y}$ , the likelihood defined by Eqs. (13) and (14) is evaluated as a function of the variable  $a = a_{\text{dum}}$  which now has to be considered as a dummy variable, distinct from  $a_{\text{true}}$ . The likelihood function is thus calculated as  $L(\mathbf{Y}, a_{\text{dum}})$ , a function of  $a_{\text{dum}}$  at fixed  $\mathbf{Y}$  given by the observation. This calculation of  $L(\mathbf{Y}, a_{\text{dum}})$  is performed over a discrete set of values for the variable  $a_{\text{dum}}$ , which is taken to vary over a finite interval  $[a_{\text{min}}, a_{\text{max}}]$  and sampled with a step  $\Delta a$ . The value of  $a_{\text{dum}}$  achieving the maximum of  $L(\mathbf{Y}, a_{\text{dum}})$  is then taken as the maximum likelihood estimate  $\hat{a}_{ML}$  of Eq. (15). The current difference  $(\hat{a}_{ML} - a_{\text{true}})^2$  is accumulated for the numerical evaluation of the rms estimation error  $\mathcal{E}$  of Eq. (16), as the whole process is iterated starting with another vector realization of the native noise. On the one hand,  $a_{\text{true}} \pm \alpha(\sigma_\xi + \sigma_\eta)$ , with  $\alpha = 3$  to 4, measures the possible range of equivocation of the estimate  $\hat{a}_{ML}$  with large probability. It is then consistent to choose the interval  $[a_{\text{min}}, a_{\text{max}}]$  sufficiently large to contain the range  $a_{\text{true}} \pm \alpha(\sigma_\xi + \sigma_\eta)$ . On the other hand,  $\hat{a}_{ML}$  is expected to have a rms error  $\mathcal{E}$  lower bounded by  $1/\sqrt{MJ_Y}$ . It is then consistent to choose a step

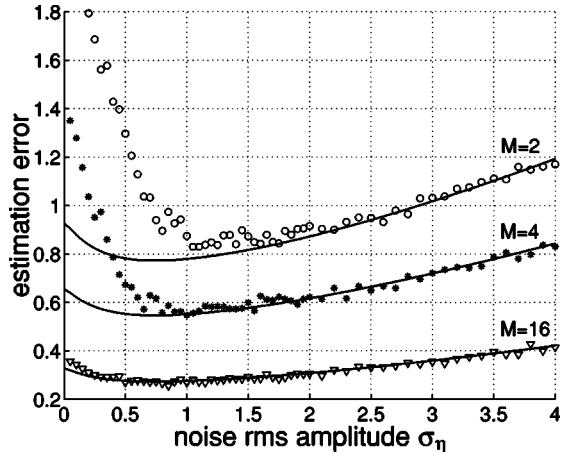


FIG. 4. Estimation with  $M$  measurements from an array of  $N = 15$  threshold devices. In abscissa is the rms amplitude  $\sigma_\eta$  of the array noises  $\eta_i(t)$  chosen zero-mean Gaussian. The discrete sets of points are the rms estimation error  $\mathcal{E}$  of Eq. (16) numerically evaluated from  $10^3$  Monte Carlo trials of the maximum likelihood estimator from Eqs. (13)–(15) for each  $\sigma_\eta$ . The solid lines are the Cramér-Rao bound  $1/\sqrt{MJ_Y}$  from Eq. (3). The input random signal  $x(t)$  is Gaussian with mean  $a=0.5$  and standard deviation  $\sigma_x=1$ . The common threshold is  $\theta=0$ .

$\Delta a$  sufficiently small compared to  $1/\sqrt{MJ_Y}$ . These two choices, statistically, will guarantee an accurate evaluation of error  $\mathcal{E}$  in the Monte Carlo procedure. These choices rely on prior informations that are usually available in conventional estimation problems, concerning the noise densities and the feasible range for the values accessible to the parameter  $a = a_{\text{true}}$  to be estimated. At least, these prior informations for configuring  $[a_{\text{min}}, a_{\text{max}}]$  and  $\Delta a$  are available under the form of conservative estimates or bounds, which is appropriate for the numerical implementation of the maximum likelihood estimator. These considerations lead us, for the conditions of Fig. 4, to choose  $a_{\text{max}} = -a_{\text{min}} = 15$  and  $\Delta a = 0.01$ . Figure 4 presents evolutions of the rms error  $\mathcal{E}$  evaluated with this Monte Carlo implementation of the maximum likelihood estimator.

The evolutions of Fig. 4 show, as expected, that as  $M$  increases, the rms estimation error  $\mathcal{E}$  reaches the Cramér-Rao bound  $1/\sqrt{MJ_Y}$  derived from the Fisher information of Eq. (3). This behavior, asymptotic in principle, is in practice well realized as soon as moderate values of  $M$ , such as  $M = 16$  in Fig. 4. This demonstrates that the possibilities of enhancement by noise of the Fisher information, as expressed by Figs. 1–3, will be precisely reflected in the performance of actual estimators like the maximum likelihood estimator, even when using moderate numbers  $M$  of measurements for estimation. Furthermore, it is visible in Fig. 4 that for small values of  $M$ , such as  $M = 2$ , even though the rms error  $\mathcal{E}$  is not quantitatively precisely described by the Fisher information (especially at low  $\sigma_\eta$ ), there still exists a similar possibility of improving the performance (the rms error here) by means of an increase in the level  $\sigma_\eta$  of the array noises  $\eta_i(t)$ . This is expressed in Fig. 4 by the rms estimation error  $\mathcal{E}$  which starts to decrease when the noise level  $\sigma_\eta$  increases, with an optimal nonzero value of  $\sigma_\eta$  that minimizes  $\mathcal{E}$ .

Moreover, it appears in the conditions of Fig. 4 that at small  $M$ , the improvement by the array noises induced on the error  $\mathcal{E}$ , is even more pronounced than the improvement expressed by the Fisher information through the Cramér-Rao bound. In other words, at small  $M$ , the suprathreshold SR is more pronounced when assessed by the rms estimation error than by the Fisher information. Qualitatively, all the evolutions of Fig. 4 point to the same idea, in one form or another: the possibility of an improvement by addition of threshold noises, of the performance of a parallel array of quantizers used for estimation on a suprathreshold signal.

Again, the conditions of Fig. 4 are merely illustrative, and the possibility of improvement by addition of array noises of the rms error  $\mathcal{E}$  is preserved in many other conditions of estimation on the suprathreshold signal  $x(t)$ . It is also expected that the more elaborate scenarios for estimation, as evoked in Secs. III and IV, will also reflect qualitatively similar possibilities of suprathreshold SR on the rms estimation error.

## VI. ESTIMATION ON TIME-VARYING SIGNALS

For estimation on an input signal with the form  $x(t) = s_a(t) + \xi(t)$ , the above equations allowing one to derive the Fisher information  $J_Y$ , are valid for any type of parametric  $s_a(t)$ , although we have used them only for a constant  $s_a(t) \equiv a$  in the examples of Figs. 1–4. Compared to the case of a constant  $s_a(t) \equiv a$ , the Fisher information for a time-varying  $s_a(t)$  only differs, at any time  $t$ , by the multiplication by the prefactor  $[\partial s_a(t)/\partial a]^2$  of Eq. (7) or (10), the constant value of  $a$  being replaced by the instantaneous value of  $s_a(t)$  at time  $t$ .

For instance, for estimation of the amplitude  $a$  of a periodic input  $s_a(t) = a \cos(\omega t)$  with known angular frequency  $\omega$ , the derivative in Eqs. (7) and (10) is  $\partial s_a(t)/\partial a = \cos(\omega t)$ . Compared to the Fisher information  $J_Y$  for a constant  $s(t) \equiv a$ , the Fisher information is obtained through the change  $J_Y \rightarrow \cos^2(\omega t) J_Y$ . We end up with a time-dependent Fisher information, which is known at every time  $t$ , with a time variation involving the prefactor  $\cos^2(\omega t)$ . At any time  $t$ , the Fisher information keeps the same significance for assessing the performance in estimation. If the measurements are taken at times  $t$  where the input  $s_a(t) = a \cos(\omega t)$  is zero, the Fisher information at these times is also zero. This means, in a quite consistent way, that there is no possibility of estimating the amplitude  $a$  of  $s_a(t)$  if data are collected when  $s_a(t)$  is zero; and on the contrary, the efficacy in estimation (and the Fisher information) will be high if data are collected at times when  $s_a(t)$  assumes large values.

As another example, for estimation of the angular frequency  $a$  of a periodic input  $s_a(t) = A \cos(at)$  with known amplitude  $A$ , the derivative in Eqs. (7) and (10) is  $\partial s_a(t)/\partial a = -At \sin(at)$ . Compared to the Fisher information  $J_Y$  for a constant  $s(t) \equiv a$ , the Fisher information is now obtained through the change  $J_Y \rightarrow [At \sin(at)]^2 J_Y$ . We again end up with a time-dependent Fisher information, which is known at every time  $t$ , with a time variation involving the prefactor  $[At \sin(at)]^2$ . Again, at any time  $t$ , the Fisher information keeps the same significance for assessing the perfor-

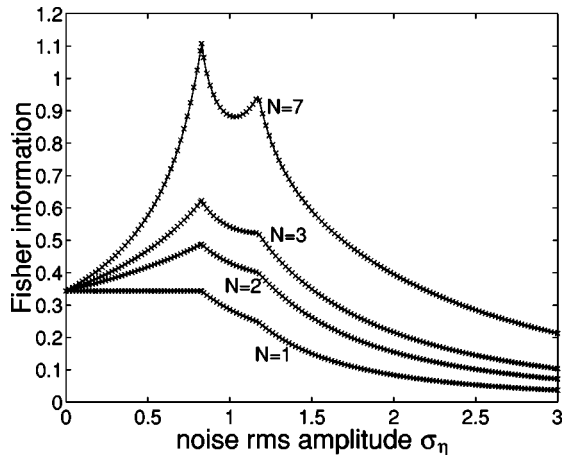


FIG. 5. Fisher information  $J_Y$  of Eq. (3), as a function of the rms amplitude  $\sigma_\eta$  of the array noises  $\eta_i(t)$  chosen zero-mean uniform. The input random signal  $x(t)$  is uniform with mean  $a=0.3$  and standard deviation  $\sigma_x=1$ . The value  $\theta=0$  is assigned to all thresholds in the array. ( $\times$ ) are  $J_Y$  computed from the analytical expressions of the Appendix. The solid lines are  $J_Y$  computed by numerical evaluation of the integrals of Eqs. (6) and (7).

mance in estimation. If the time origin is arbitrarily taken, the prefactor  $[At \sin(at)]^2$ , and hence the Fisher information, can grow without bound. In such a case, the Cramér-Rao bound becomes a vanishingly small lower bound, suggesting that the variance of any unbiased estimator of  $a$  is necessarily above zero. This is a consistent, although not very informative, deduction. With another time origin, chosen at the time of measurement, i.e.,  $t=0$ , we make the prefactor  $[At \sin(at)]^2$ , and hence the Fisher information, zero. In such a case, the Cramér-Rao bound becomes infinite, suggesting that the variance of any unbiased estimator of  $a$  is also infinite. This is again a consistent statement, revealing that there is no possibility of estimating the frequency of a periodic signal from measurements taken at a single fixed time. More than one time are necessary to pick up the measurements, at which times the Fisher information cannot be simultaneously zero [unless those times are exactly located at the zeros of  $\sin(at)$ ], this being associated to a noninfinite Cramér-Rao bound allowing a noninfinite variance of the estimators.

However, it is important to realize that those temporal behaviors of the Fisher information with a time-varying signal  $s_a(t)$  are caused by the very nature of the time-varying input, and are not specifically related to the array of threshold devices we use for estimation. For instance with a periodic signal  $s_a(t)$ , the temporal behavior would be much similar to the Fisher information for estimation based directly on the input signal  $s_a(t) + \xi(t)$ . In such a case also, the Fisher information is affected in the same way by the prefactor  $[\partial s_a(t)/\partial a]^2$ , inherent to the definition of the Fisher information involving a derivative in the parameter.

The specificity of the results we present here are not contained in the temporal evolution of the Fisher information, but they are contained in its evolution with the level of the noise. In the expression of the Fisher information  $J_Y$ , both the signal  $s_a(t)$  and the prefactor  $\partial s_a(t)/\partial a$  in Eqs. (7) and (10) are independent of the array noises  $\eta_i(t)$  and of the

native noise  $\xi(t)$ . Thus, for the influence of the noise properties on the Fisher information of Eq. (3), the specific form of the parametric dependence of  $s_a(t)$  on  $a$  will play no role since it does not vary with the noise. As a consequence, any type of signal  $s_a(t)$  parametrized in any arbitrary way by  $a$  will benefit from the same possibility of improvement by noise as expressed by Figs. 1–3 on the Fisher information. In this respect, the evolutions with the noise level obtained in Figs. 1–3 for a constant signal  $s_a(t)$  are in some sense generic and they bear significance for any type of parametric  $s_a(t)$ .

## VII. DISCUSSION

We have demonstrated the feasibility of suprathreshold SR for parametric estimation performed from the output of a parallel array of threshold devices. The results reported here aim essentially at proving the feasibility, in principle, of a suprathreshold SR in parametric estimation from an array of threshold devices. This is realized here in several representative conditions. These conditions are not in themselves critical for the observation of suprathreshold SR. Beyond this, many aspects of this form of suprathreshold SR remain open for future investigation, for example, further analysis of the influence of the type of the probability density of the threshold noises or the development of adaptive schemes to apply the right amount of threshold noise to definite estimators.

The suprathreshold SR demonstrated here, similar to that in Refs. [11,12], is especially operative when the thresholds in the array are constrained to be the same and cannot be separately adjusted. Such conditions can be encountered in natural systems such as neurons organized in parallel arrays for sensory processing. A form of suprathreshold SR measured by the input-output mutual information has been shown possible in arrays of sensory neurons [23]. It is likely that the present form of suprathreshold SR measured by the Fisher information can also take place in neuronal arrays in charge of estimation tasks in sensory processing.

More generally, suprathreshold SR now gradually emerges as a mechanism of improvement by noise, with general significance and applicability envisagable under many different forms, even going beyond the special instances assessed through the Fisher or the Shannon information we discussed here. In broad terms, standard (subthreshold) SR can be described as assistance brought by noise, to a small signal, in eliciting a response from a single nonlinear system. By contrast, suprathreshold SR can be described as diversity brought by noise for a richer response, to a suprathreshold signal, elicited by an array of nonlinear systems. These constitute two distinct forms of improvement by noise, with rich potentialities still to be explored for nonlinear information processing.

## ACKNOWLEDGMENT

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## APPENDIX

In this appendix, we consider the case of uniform noises  $\xi(t)$  and  $\eta_i(t)$ , and equal thresholds  $\theta_i = \theta$ ,  $\forall i$ . These conditions allow us to obtain an analytical expression for the Fisher information  $J_Y$  of Eq. (3), thanks to analytical integration of the integrals of Eqs. (6) and (7). This analytical treatment of the Fisher information parallels that of Refs. [17] for the Shannon information.

Let

$$Q(n, x) = C_n^N [1 - q(x)]^n q(x)^{N-n}. \quad (\text{A1})$$

When the native noise  $\xi(t)$  is uniform over  $[-\sqrt{3}\sigma_\xi, \sqrt{3}\sigma_\xi]$ , the probability density  $f_x(x) = f_\xi[x - s_a(t)]$  takes the constant value  $1/(2\sqrt{3}\sigma_\xi)$  when  $x \in [-\sqrt{3}\sigma_\xi + s_a(t), \sqrt{3}\sigma_\xi + s_a(t)]$ , and is zero outside this interval. Therefore, Eq. (6) transforms into

$$\Pr\{Y(t) = n\} = \frac{1}{2\sqrt{3}\sigma_\xi} \int_{-\sqrt{3}\sigma_\xi + s_a(t)}^{\sqrt{3}\sigma_\xi + s_a(t)} Q(n, x) dx. \quad (\text{A2})$$

We also have

$$Q(n, x) = \begin{cases} C_n^N \left( -\frac{\theta - x - \sqrt{3}\sigma_\eta}{2\sqrt{3}\sigma_\eta} \right)^n \left( \frac{\theta - x + \sqrt{3}\sigma_\eta}{2\sqrt{3}\sigma_\eta} \right)^{N-n} & \text{for } \theta - \sqrt{3}\sigma_\eta \leq x \leq \theta + \sqrt{3}\sigma_\eta \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A6})$$

Also

$$Q(0, x) = \begin{cases} 1 & \text{for } x < \theta - \sqrt{3}\sigma_\eta \\ \left( \frac{\theta - x + \sqrt{3}\sigma_\eta}{2\sqrt{3}\sigma_\eta} \right)^N & \text{for } \theta - \sqrt{3}\sigma_\eta \leq x \leq \theta + \sqrt{3}\sigma_\eta \\ 0 & \text{for } x > \theta + \sqrt{3}\sigma_\eta \end{cases} \quad (\text{A7})$$

and

$$Q(N, x) = \begin{cases} 0 & \text{for } x < \theta - \sqrt{3}\sigma_\eta \\ \left( -\frac{\theta - x - \sqrt{3}\sigma_\eta}{2\sqrt{3}\sigma_\eta} \right)^N & \text{for } \theta - \sqrt{3}\sigma_\eta \leq x \leq \theta + \sqrt{3}\sigma_\eta \\ 1 & \text{for } x > \theta + \sqrt{3}\sigma_\eta. \end{cases} \quad (\text{A8})$$

We introduce  $x_{\text{inf}} = \max[-\sqrt{3}\sigma_\xi + s_a(t), \theta - \sqrt{3}\sigma_\eta]$  and  $x_{\text{sup}} = \min[\sqrt{3}\sigma_\xi + s_a(t), \theta + \sqrt{3}\sigma_\eta]$ . For any  $n \in [1, N-1]$ , Eq. (A2) yields  $\Pr\{Y(t) = n\} = C_n^N I(n) \sigma_\eta / \sigma_\xi$ , with the integral

$$I(n) = \frac{1}{2\sqrt{3}\sigma_\eta} \int_{x_{\text{inf}}}^{x_{\text{sup}}} \left( -\frac{\theta - x - \sqrt{3}\sigma_\eta}{2\sqrt{3}\sigma_\eta} \right)^n \times \left( \frac{\theta - x + \sqrt{3}\sigma_\eta}{2\sqrt{3}\sigma_\eta} \right)^{N-n} dx. \quad (\text{A9})$$

$$f'_\xi[x - s_a(t)] = \frac{1}{2\sqrt{3}\sigma_\xi} \{ \delta[x + \sqrt{3}\sigma_\xi - s_a(t)] - \delta[x - \sqrt{3}\sigma_\xi - s_a(t)] \}, \quad (\text{A3})$$

so Eq. (7) becomes

$$\frac{\partial}{\partial a} \Pr\{Y(t) = n\} = -\frac{\partial s_a(t)}{\partial a} \frac{1}{2\sqrt{3}\sigma_\xi} \{ Q[n, -\sqrt{3}\sigma_\xi + s_a(t)] - Q[n, \sqrt{3}\sigma_\xi + s_a(t)] \}. \quad (\text{A4})$$

We further assume that the threshold noises  $\eta_i(t)$  are uniform over  $[-\sqrt{3}\sigma_\eta, \sqrt{3}\sigma_\eta]$ , with the cumulative distribution function

$$F_\eta(u) = \begin{cases} 0 & \text{for } u < -\sqrt{3}\sigma_\eta \\ \frac{1}{2\sqrt{3}\sigma_\eta} (u + \sqrt{3}\sigma_\eta) & \text{for } -\sqrt{3}\sigma_\eta \leq u \leq \sqrt{3}\sigma_\eta \\ 1 & \text{for } u > \sqrt{3}\sigma_\eta. \end{cases} \quad (\text{A5})$$

Since  $q(x) = F_\eta(\theta - x)$ , Eq. (A1) becomes, for any integer  $n \in [1, N-1]$ ,

Through the change of variable  $z = (\theta - x + \sqrt{3}\sigma_\eta)/(2\sqrt{3}\sigma_\eta)$ , we get

$$I(n) = \int_{z_{\text{inf}}}^{z_{\text{sup}}} (1-z)^n z^{N-n} dz, \quad (\text{A10})$$

with

$$z_{\text{inf}} = \max \left[ 0, \frac{1}{2} \left( 1 - \frac{\sigma_\xi}{\sigma_\eta} + \frac{\theta - s_a(t)}{\sqrt{3}\sigma_\eta} \right) \right], \quad (\text{A11})$$



$$z_{\text{sup}} = \min \left[ 1, \frac{1}{2} \left( 1 + \frac{\sigma_\xi}{\sigma_\eta} + \frac{\theta - s_a(t)}{\sqrt{3}\sigma_\xi} \right) \right]. \quad (\text{A12})$$

The integral of Eq. (A10) can generally be evaluated by means of the primitive  $\int (1-z)^n z^{N-n} dz = {}_{N+2}F_1([N-n+1, -n], [N-n+2], z) z^{N-n+1}/(N-n+1)$ , where  ${}_mF_p$  is the generalized hypergeometric function or Barnes's extended hypergeometric function [24,25]. An alternative possibility to integrate  $I(n)$  of Eq. (A10) is to use the binomial expansion of  $(1-z)^n$  to obtain

$$\int (1-z)^n z^{N-n} dz = \sum_{k=0}^n (-1)^k C_k^n \frac{z^{N-(n-k)+1}}{N-(n-k)+1}, \quad (\text{A13})$$

which replaces  $I(n)$  by a finite sum, for any  $n \in [0, N]$ . This last integral of Eq. (A13) simplifies into  $z^{N+1}/(N+1)$  for  $n=0$  and into  $-(1-z)^{N+1}/(N+1)$  for  $n=N$ .

As written above, knowledge of  $I(n)$  provides access to  $\Pr\{Y(t)=n\} = C_n^N I(n) \sigma_\eta / \sigma_\xi$ , for any  $n \in [1, N-1]$ . In addition, we have from Eqs. (A2) and (A7),

$$\Pr\{Y(t)=0\} = \begin{cases} I(0) \frac{\sigma_\eta}{\sigma_\xi} + \frac{1}{2} \left( 1 - \frac{\sigma_\eta}{\sigma_\xi} + \frac{\theta - s_a(t)}{\sqrt{3}\sigma_\xi} \right) & \text{for } \sigma_\eta < \sigma_\xi + \frac{\theta - s_a(t)}{\sqrt{3}} \\ I(0) \frac{\sigma_\eta}{\sigma_\xi} & \text{otherwise} \end{cases} \quad (\text{A14})$$

and from Eqs. (A2) and (A8),

$$\Pr\{Y(t)=N\} = \begin{cases} I(N) \frac{\sigma_\eta}{\sigma_\xi} + \frac{1}{2} \left( 1 - \frac{\sigma_\eta}{\sigma_\xi} - \frac{\theta - s_a(t)}{\sqrt{3}\sigma_\xi} \right) & \text{for } \sigma_\eta < \sigma_\xi - \frac{\theta - s_a(t)}{\sqrt{3}} \\ I(N) \frac{\sigma_\eta}{\sigma_\xi} & \text{otherwise.} \end{cases} \quad (\text{A15})$$

By collecting the equations of this appendix, we now have the possibility of a direct evaluation of the Fisher information  $J_Y$  of Eq. (3) avoiding numerical approximations of integrals, in the case of uniform noises  $\xi(t)$  and  $\eta_i(t)$ , in general conditions concerning  $s_a(t)$  and  $N$ . An illustration is provided in Fig. 5. Beyond the specificities of the uniform noises [with the bounded support of the noises  $\eta_i(t)$  which, as  $\sigma_\eta$  grows, crosses critical levels related to the support of the native noise  $\xi(t)$ , to the signal amplitude  $s_a(t)$  and to the threshold  $\theta$ ], the important observation we want to emphasize in Fig. 5 is that the suprathreshold SR is preserved qualitatively in a quite similar form as with Gaussian noises. Noise enhancement of the Fisher information occurs for a

suprathreshold signal, as soon as  $N > 1$ . This demonstrates a robustness, or universality, of the qualitative features that define suprathreshold SR, with respect to the distribution of the noises.

Further simplifications of the analytical expressions for  $J_Y$  can be obtained in more specific configurations. When  $\sigma_\eta < \sigma_\xi - [\theta - s_a(t)]/\sqrt{3}$ , then Eq. (A11) yields  $z_{\text{inf}}=0$ ; when  $\sigma_\eta < \sigma_\xi + [\theta - s_a(t)]/\sqrt{3}$ , then Eq. (A12) yields  $z_{\text{sup}}=1$ . With  $z_{\text{inf}}=0$  and  $z_{\text{sup}}=1$  in Eq. (A10), the generalized hypergeometric function simplifies into the Beta function and Eq. (A10) gives  $I(n) = n!(N-n)!/(N+1)!$  for any  $n \in [0, N]$ . This leads Eq. (A2) to

$$\Pr\{Y(t)=n\} = \begin{cases} \left( \frac{1}{N+1} - \frac{1}{2} \right) \frac{\sigma_\eta}{\sigma_\xi} + \frac{1}{2} \left( 1 + \frac{\theta - s_a(t)}{\sqrt{3}\sigma_\xi} \right) & \text{for } n=0 \\ \frac{1}{N+1} \frac{\sigma_\eta}{\sigma_\xi} & \text{for } 0 < n < N \\ \left( \frac{1}{N+1} - \frac{1}{2} \right) \frac{\sigma_\eta}{\sigma_\xi} + \frac{1}{2} \left( 1 - \frac{\theta - s_a(t)}{\sqrt{3}\sigma_\xi} \right) & \text{for } n=N. \end{cases} \quad (\text{A16})$$

Also, when the same two conditions are met, i.e., when  $\sigma_\eta < \sigma_\xi - |\theta - s_a(t)|/\sqrt{3}$ , then Eq. (A4) gives

$$\frac{\partial}{\partial a} \Pr\{Y(t)=n\} = \begin{cases} -\frac{\partial s_a(t)}{\partial a} \frac{1}{2\sqrt{3}\sigma_\xi} & \text{for } n=0 \\ 0 & \text{for } 0 < n < N \\ \frac{\partial s_a(t)}{\partial a} \frac{1}{2\sqrt{3}\sigma_\xi} & \text{for } n=N. \end{cases} \quad (\text{A17})$$

This leads to an explicit expression for the Fisher information  $J_Y$  of Eq. (3), valid for  $\sigma_\eta < \sigma_\xi - |\theta - s_a(t)|/\sqrt{3}$ , as

$$J_Y = \left[ \frac{\partial s_a(t)}{\partial a} \right]^2 \left( \frac{1}{2\sqrt{3}\sigma_\xi} \right)^2 \left[ \frac{1}{\left( \frac{1}{N+1} - \frac{1}{2} \right) \frac{\sigma_\eta}{\sigma_\xi} + \frac{1}{2} \left( 1 + \frac{\theta - s_a(t)}{\sqrt{3}\sigma_\xi} \right)} + \frac{1}{\left( \frac{1}{N+1} - \frac{1}{2} \right) \frac{\sigma_\eta}{\sigma_\xi} + \frac{1}{2} \left( 1 - \frac{\theta - s_a(t)}{\sqrt{3}\sigma_\xi} \right)} \right]. \quad (\text{A18})$$

This expression of  $J_Y$  in Eq. (A18), over its domain of validity, for any  $N > 1$ , is a strictly increasing function of  $\sigma_\eta$  when  $\sigma_\eta$  starts to grow above zero in the presence of  $\sigma_\xi > 0$ ; by contrast, it is a constant function of  $\sigma_\eta$  when  $N = 1$ . This is a direct proof of the suprathreshold SR effect in the array: when threshold noise is added in the array ( $N > 1$ ), the Fisher information  $J_Y$  starts to grow, while this growth is not present with a single device ( $N = 1$ ).

In addition, the above equations for  $J_Y$  with uniform noises  $\xi(t)$  and  $\eta_i(t)$  show that the optimal level of the threshold noises which maximizes the Fisher information is  $\sigma_\eta = \sigma_\xi - |\theta - s_a(t)|/\sqrt{3}$ , as seen in Fig. 5. This holds when  $\sigma_\xi > |\theta - s_a(t)|/\sqrt{3}$ ; otherwise, we are outside the regime of suprathreshold SR which is of interest to us here, because then the input signal  $x(t) = s_a(t) + \xi(t)$  has become subthreshold when the threshold noises  $\eta_i(t)$  are absent.

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- [1] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, *Rev. Mod. Phys.* **70**, 223 (1998).
- [2] J.J. Collins, C.C. Chow, and T.T. Imhoff, *Phys. Rev. E* **52**, R3321 (1995).
- [3] Z. Gingl, L.B. Kiss, and F. Moss, *Europhys. Lett.* **29**, 191 (1995).
- [4] M.E. Inchiosa and A.R. Bulsara, *Phys. Rev. E* **53**, R2021 (1996).
- [5] F. Chapeau-Blondeau and X. Godivier, *Phys. Rev. E* **55**, 1478 (1997).
- [6] F. Chapeau-Blondeau, *Phys. Rev. E* **55**, 2016 (1997).
- [7] S. Mitaïm and B. Kosko, *Proc. IEEE* **86**, 2152 (1998).
- [8] D.G. Luchinsky, R. Mannella, P.V.E. McClintock, and N.G. Stocks, *IEEE Trans. Circuits Syst., II: Analog Digital Signal Process.* **46**, 1205 (1999).
- [9] P.O. Amblard and S. Zozor, *Phys. Rev. E* **59**, 5009 (1999).
- [10] G.P. Harmer, B.R. Davis, and D. Abbott, *IEEE Trans. Instrum. Meas.* **51**, 299–309 (2002).
- [11] N.G. Stocks, *Phys. Rev. Lett.* **84**, 2310 (2000).
- [12] N.G. Stocks, *Phys. Rev. E* **63**, 041114 (2001).
- [13] M. Stemmler, *Network Comput. Neural Syst.* **7**, 687 (1996).
- [14] H.C. Papadopoulos, G.W. Wornell, and A.V. Oppenheim, in *Proceedings of IEEE International Conference on Acoustics, Speech and Signal Processing* (IEEE Press, Piscataway, 1998), pp. 3273–3276.
- [15] P.E. Greenwood, L.M. Ward, and W. Wefelmeyer, *Phys. Rev. E* **60**, 4687 (1999).
- [16] T.M. Cover and J.A. Thomas, *Elements of Information Theory* (Wiley, New York 1991).
- [17] N.G. Stocks, *Phys. Lett. A* **279**, 308 (2001).
- [18] T. Hoch, G. Wenning, and K. Obermayer, *Neurocomputing*, **52–54**, 541 (2003).
- [19] M.D. McDonnell, D. Abbott, and C.E.M. Pearce, *Microelectron. J.* **33**, 1079 (2002).
- [20] G. Wenning and K. Obermayer, *Phys. Rev. Lett.* **90**, 120602 (2003).
- [21] D. Rousseau, G.V. Anand, and F. Chapeau-Blondeau, in *Proceedings of the Third International Symposium on Physics in Signal and Image Processing*, Grenoble, France, 2003 (SEE, Paris, 2003), pp. 89–92.
- [22] H.V. Poor, *An Introduction to Signal Detection and Estimation* (Springer, Berlin 1994).
- [23] N.G. Stocks and R. Mannella, *Phys. Rev. E* **64**, 030902 (2001).
- [24] D. Redfern, *Maple V Handbook* (Springer-Verlag, Berlin 1997).
- [25] E.W. Weisstein, *CRC Concise Encyclopedia of Mathematics* (CRC Press, Boca Raton, FL, 1999).